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Observability by using viability kernels

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Abstract: The aim of this paper is to propose a new way to deal with observability of systems governed by ODEs, in a more general setting than the standard output equation. The primary finding is that observability over a time horizon reduces to single-valuedness of the vertical section of a set we name the observability kernel. The latter consists of the viability kernel of the output domain under the augmented system. The approach may be used either for global or local observability, to which available results on single-valuedness of multifunctions shall be applied in order to get necessary and/or sufficient characterizing conditions. Several examples are provided in order to illustrate the method.

Keywords: Observability; viability kernel; single-valuedness; lower semi-continuity; monotone multifunctions

1 Introduction and motivation

Observability is one of the old concepts of control theory. It addresses the issue of whether the initial state of a system can be uniquely determined, out of the information provided by the current output.

The linear case, either in finite or infinite dimension, has been thoroughly investigated and characterizing global results obtained. In contrast, nonlinear observability remains largely unexplored, and most of the existing results [1∼6] derived by the geometric differential approach are of local character.

We assign the objective of this note to provide an alternative set-valued approach within a more general framework in regard to the output expression, and which can be introduced in the following.

Let n be an integer and $f:\mathbb{R}_+\times\mathbb{R}^n\to\mathbb{R}^n$ be a *continuous* function. For $t_f>0$, consider the system governed by the ordinary differential equation,

$$\dot z(t)=f(t,z(t)),\ \ t\in[t_0,t_f], \tag{1a}$$

with output expressed as follows,

$$(t,z(t))\in\Theta,\ \ t\in[t_0,t_f], \tag{1b}$$

where $t_0\in[0,t_f)$ and Θ is a *closed* subset of $\mathbb{R}_+\times\mathbb{R}^n$, which we call the *output domain*.

One immediately can see that this new setting encompasses the standard situation in which both observation θ and state z satisfy the output equation,

$$\theta(t)=h(t,z(t)),\ \ t\in[t_0,t_f], \tag{2}$$

for some smooth function h. For this, it is obvious that the output domain can be given as follows,

$$\Theta\doteq\{(t,z)\in\mathbb{R}_+\times\mathbb{R}^n\mid h(t,z)-\theta(t)=0\}. \tag{3}$$

Moreover, Eq. (1b) may serve to describe incomplete, partial or uncertain informations on the state of the system. For instance, whenever uncertainties are assumed to affect the measurements in Eq. (2) then the output domain can be expressed in the form,

$$\Theta\doteq\{(t,z)\in\mathbb{R}_+\times\mathbb{R}^n\mid |h(t,z)-\theta(t)|\le\epsilon(t,z)\}, \tag{4}$$

for a positive function ϵ defined on $\mathbb{R}_+\times\mathbb{R}^n$.

The preceding facts lead us to restate the usual issue of observability concept as whether the initial state can be *uniquely* reconstructed on the basis of :

(α) information on the current state of the system, here expressed in the general form of Eq. (1b).

(β) a primary knowledge that the initial state belongs to a subset $\Sigma\subset\mathbb{R}^n$. In fact, often it may happen that the initial states which are susceptible to generate the observed information or output are known to belong to a given subset.

Following are the classical statements of observability, which we adapt in light of the above motivational ideas.

Definition 1 Let $t_0\in[0,t_f)$, z_0, z_1 and z_2 belong to $\mathbb{R}^n$ and Σ be a subset of $\mathbb{R}^n$.

(a) We say that state z_0 generates output (1b) on the horizon $[t_0,t_f]$, if the constrained ODE (1) has a solution $\bar z$ which satisfies $\bar z(t_0)=z_0$. For notation :

$$z_0\overset{f}{\leadsto}\Theta \text{ on } [t_0,t_f].$$

(b) The states z_1 and z_2 are said to be indistinguishable on $[t_0,t_f]$ if both generate the output (1b) on the interval $[t_0,t_f]$.

(c) System (1) is said to be $\Sigma-$observable on the horizon $[t_0,t_f]$ if there are no pairs of distinct indistinguishable states on $[t_0,t_f]$, which are included in subset Σ.

(d) System (1) is said to be observable on the horizon $[t_0,t_f]$ if it is $\mathbb{R}^n-$observable on $[t_0,t_f]$.

(e) System (1) is said to be locally observable on the horizon $[t_0,t_f]$ around state z_0, if there exists $W\in\mathcal{N}(z_0)$ such that system (1) is $W-$observable on $[t_0,t_f]$. Here

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$\mathcal{N}(z)$ is used to denote the collection of neighborhoods of a point z in a metric space.

In this context, we consider an additional concept of local observability with respect to time.

Definition 2 For $\bar{t} \in (0, t_f)$, System (1) is said to be observable from near $\bar{t}$ if there exists $\mathcal{J} \in \mathcal{N}(\bar{t})$ such that for all $t_0 \in \mathcal{J}$, System (1) is observable on $[t_0, t_f]$. It is said to be continuously observable from near $\bar{t}$ if, further, the function $t_0 \in \mathcal{J} \to z_0$, (where $z_0 \stackrel{f}{\leadsto} \Theta$ on $[t_0, t_f]$) is well defined and continuous.

Note the concept of localness we have introduced in Definition (2) above is rather viewed with respect to starting time. The issue may therefore be of particular interest whenever the initial time of the observation process is known to be uncertain.

Through this paper the following notations will be considered : For an Euclidean space, the inner product is denoted by $\langle \cdot, \cdot \rangle$, the corresponding norm by $|\cdot|$ and the diameter of a subset K by

$$\mathrm{diam}(K) \doteq \sup\{|x - y| \mid x, y \in K\}.$$

While, we will denote by $\mathrm{card}(E)$ the cardinal number of a finite set E. Let F stand for a multifunction, we denote its domain by

$$\mathrm{dom}(F) \doteq \{x \mid F(x) \neq \emptyset\},$$

and its graph by

$$\mathrm{gph}(F) \doteq \{(x, y) \mid y \in F(x)\}.$$

Also we will use the notation,

$$F^{-1}(y) \doteq \{x \mid y \in F(x)\},$$

to define the inverse of multifunction F. Finally we say that a property holds near a point iff it holds for all points of a neighborhood of that point.

The organization of this paper is described as follows. In section 2 we state preliminary definitions and results. The aim of section 3 is to show necessary and/or sufficient conditions for observability. The latter is studied in section 4, in connection with concepts of single-valuedness and monotonicity of multifunctions.

2 Definitions and preliminary results

This section is devoted to transferring some definitions and basic results which are used through the paper. These involve notions of viability kernel and single-valuedness of multifunctions.

2.1 The viability kernel

Viability theory [7] stands for an adequate framework to study differential inclusions with constraints. Its main foundations can be outlined in the following.

A subset K is said to be (locally) viable under a multifunction F if for all $z_0 \in K$, the constrained differential inclusion,

$$\dot{z} \in F(z), \ z \in K, \tag{5}$$

has a solution which is issued from z_0. Recall [7] that when F is upper semi-continuous (ie. $\{x \mid F(x) \cap C \neq \emptyset\}$) is closed for each closed subset C) and K is locally compact then a necessary and sufficient condition in order that K be viable under F is provided by the following tangential condition,

$$F(z) \cap T_K(z) \neq \emptyset,$$

for all $z \in K$, where $T_K(\cdot)$ denotes the cotangent cone, given for all $z \in K$ by,

$$T_K(z) \doteq \{y \in \mathbb{R}^n \mid \lim_{h \downarrow 0} \frac{d(z + hy, K)}{h} = 0\}.$$

Assume that K is closed and let $t_f > 0$. The t_f-viability kernel [7, 8] of subset K under F is the set of all initial states $z_0 \in K$ for which the differential inclusion (5) has a solution on $[0, t_f]$. It is denoted by $\mathrm{viab}_F(K, t_f)$. The viability kernel of subset K under F is $\mathrm{viab}_F(K, \infty)$ and is denoted $\mathrm{viab}_F(K)$. Some of the properties of the viability kernel and remarks can be listed as follows; see [8] for deep details :

(i) When K and $\mathrm{gph}(F)$ are convex, so are the viability kernels $\mathrm{viab}_F(K)$ and $\mathrm{viab}_F(K, t_f)$ for all $t_f > 0$.

(ii) For a Marchaud [7] map F, the viability kernel coincides with the largest closed viable subset contained in K. A Marchaud map is characterized by the following statements :
  (a) It has convex values.
  (b) Its graph and its domain are nonempty and closed.
  (c) $\sup_{v \in F(x)} |v| \leq c(|x|+1)$ for all $x \in K$ for some $c > 0$.

(iii) Concerning the computational aspects, we quote the work [9] which has initiated a series of studies [10∼12] which take on algorithmic approaches to approximate the viability kernels.

2.2 When is a multifunction single-valued?

Among several works on single-valuedness of set-valued maps, the results below prove very suitable for use in the context of this paper. We divide them into two groups of results, depending on whether they are local or global.

2.2.1 Global results

The first result mainly uses algebraic properties of mid-convex multifunctions. We say that multifunction F is mid-convex if its domain is convex and it satisfies,

$$\frac{F(x_1) + F(x_2)}{2} \subset F(\frac{x_1 + x_2}{2}).$$

for all $x_1, x_2 \in \mathrm{dom}(F)$. That enables us to state the following.

Theorem 1 (Nikodem et al. [13]) Suppose that $\mathrm{dom}(F)$ is open and convex and that $F(x_0)$ is a singleton for some x_0, then F is mid-convex iff there exist an additive function $u : X \to Y$ and a constant $c \in Y$ such that,

$$F(x) = \{u(x) + c\}, \text{ for all } x \in \mathrm{dom}(F).$$

We also refer to [14∼16] for further results in the context of Theorem 1.

In another hand, monotone multifunctions have received much interest because of their ubiquitousness in variational analysis, and numerous works are devoted to their single-valuedness. For our use, we will consider two results [17, 18] which we adapt to fit into our framework. To that end we first need introduce monotone multifunctions.

In the remainder of this section we assume that $X = Y$, we say that F is monotone if,

$$\langle x_2 - x_1, y_2 - y_1 \rangle \geq 0$$

holds for all pairs (x_1, y_1) and (x_2, y_2) in $\mathrm{gph}(F)$.

Theorem 2 (Zarantonello [18]) Suppose that $F : X \to X$ is monotone and the interior of its domain is nonempty, then the set of points for which F is not single-valued has Lebesgue measure zero.

2.2.2 Local single-valuedness

The next result provides a deeper characterization of the local single-valuedness and continuity of multifunctions in terms of their lower semi-continuity and premonotonicity.

Theorem 3 (Levy-Poliquin [17]) Let $(\bar{x}, \bar{y}) \in \mathrm{gph}(F)$, the following are equivalent,

(a) F is single-valued and continuous near $\bar{x}$.

(b) F is premonotone and lower semi-continuous near $(\bar{x}, \bar{y})$.

In assertion (b) above, by *premonotone near* $(\bar{x}, \bar{y})$ it is meant that there is a neighborhood $U \times V \in \mathcal{N}((\bar{x}, \bar{y}))$ and a continuous function $\tau : U \to V$ which satisfies the inequality,

$$\langle x_2 - x_1, y_2 - y_1 \rangle \geq -\langle x_2 - x_1, \tau(x_2) - \tau(x_1) \rangle$$

holds for all pairs (x_1, y_1) and (x_2, y_2) in $\mathrm{gph}(F) \cap U \times V$.

As regards *lower semi-continuity* [19], we say that F is lower semi-continuous at $(\bar{x}, \bar{y})$ if for every sequence $(x_m)_m$ which converges to $\bar{x}$, there exists a sequence $(y_m)_m$ which satisfies both $y_m \in F(x_m)$ for all m and $y_m \to \bar{y}$.

3 Connection to viability kernels

The aim of this section is both to showcase an immediate result and to provide some illustrative examples. We first need to introduce the following subsets,

$$\kappa(\Theta, t_f) \doteq \{(t, z) \in \Theta \mid z \overset{f}{\rightsquigarrow} \Theta \text{ on } [t, t_f]\}, \qquad (6)$$

and,

$$\kappa(\Theta) \doteq \kappa(\Theta, \infty), \qquad (7)$$

Definition 3 We respectively call the subsets given by Eqs. (6) and (7), the observability kernel on the horizon $[0\ t_f]$ and the observability kernel of system (1).

For each $t \in [0, t_f)$, the set of all initial states which generate the output (1b) on $[t, t_f]$ can then be expressed by,

$$I(t, t_f) \doteq \{z \mid (t, z) \in \kappa(\Theta, t_f)\}, \qquad (8)$$

Note that the observability kernels as defined in (6) and (7) may be empty, and even if they do not, it may happen that the multifunction $I(\cdot, t_f)$ of (8) has empty values, as illustrated in Figure 1.

Next, we consider the following assumption,

$$\exists\, c > 0 \text{ such that } |f(t, z)| \leq 1 + c|(t, z)|. \qquad (9)$$

for all $(t, z) \in \Theta$. Also we set,

$$g(t, z) = (1, f(t, z))' \text{ for all } (t, z) \in \mathbb{R}_+ \times \mathbb{R}^n. \qquad (10)$$

Then, we are ready to prove the following result,

Theorem 4 Let $t_0 \in [0, t_f)$ and Σ be a subset of $\mathbb{R}^n$, then the following statements hold,

(a) $\kappa(\Theta, t_f) = \mathrm{viab}_g(\Theta, t_f)$ and $\kappa(\Theta) = \mathrm{viab}_g(\Theta)$.

(b) Under condition (9) one has for all $t_f > 0$,

$$\kappa(\Theta, t_f) = \mathrm{viab}_g(\Theta).$$

(c) System (1) is $\Sigma-$observable on $[t_0, t_f]$ iff

$$\mathrm{card}(\Sigma \cap I(t_0, t_f)) \leq 1.$$

Proof Assertion (a) is obviously due to the following remark :

$$\bar{z} \text{ satisfies (1)} \iff \begin{array}{l} t \to (t, \bar{z}(t)) \text{ is viable} \\ \text{in } \Theta \text{ with respect to } g. \end{array}$$

As regards statement (b) we refer to [7, Theorem 3.3.5] which enables us to infer that any local viable solution of system $\dot{\xi} = g(\xi)$, can be extended to a global viable solution, thanks to linear growth condition (9).

Now (c) merely follows from considering Definition 1 (c).

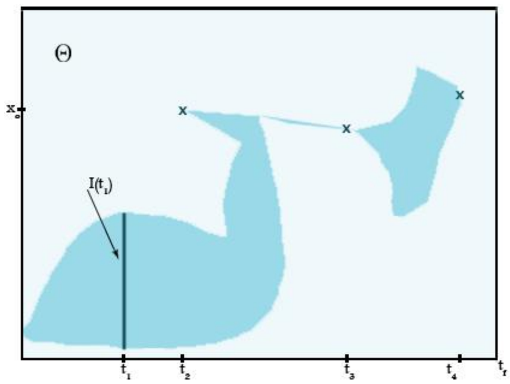

Fig. 1 A possible representation (in dark) of the observability kernel on the horizon $[0\ t_f]$: here note the system is observable on both $[0\ t_f]$ and on all the horizons $[t\ t_f]$ for $t > t_4$. It is observable from near t_3 and locally observable on $[t_2\ t_f]$ around x_0. But it is unobservable on $[t_1\ t_f]$.

From Theorem 4, we draw the following conclusions :

(1) A direct consequence is that observability over an horizon which starts from t_0 reduces to at most single-valuedness of the t_0-section of the observability kernel. Thereby all of the newly introduced concepts of observability can be checked by means of the *radioscopy* of the observability kernel, as illustrated in the imagined instance of Figure 1.

(2) It is also noteworthy to remark that when subset Θ is distinct from a singleton and is viable under the field g on the horizon $[t_0\ t_f]$, then $\kappa(\Theta, t_f) = \Theta$ and thereby the system is unobservable on any horizon $[t_0\ t_f]$: the multifunction $I(\cdot, t_f)$ of Eq. (8) is nowhere single-valued, in disagreement with the statement (c).

(3) Being in the spirit of Definition 1 (c), we notice that emptiness of subset $\Sigma \cap I(t_0, t_f)$ also involves the

$\Sigma-$observability of system (1) on $[t_0\ t_1]$. As illustrated by figure 1 for $t_0 \in (t_4\ t_f)$.

(4) For $t_1 \leq t_2$, we merely get $\kappa(\Theta, t_2) \subset \kappa(\Theta, t_1)$ and thereby $I(\cdot, t_2) \subset I(\cdot, t_1)$. It follows the natural conclusion that if the system is observable on $[t_0\ t_1]$ then it is so on $[t_0\ t_2]$.

(5) Also it must be noted that for t_f fixed, observability on $[t_0\ t_f]$ does not imply necessarily observability on $[t_1\ t_f]$ for $t_1 > t_0$, and vice-versa; see for instance Figure 2.

(6) Even, whenever $\Theta_1 \subset \Theta_2$, then one easily can see that $\kappa(\Theta_1, t_f) \subset \kappa(\Theta_2, t_f)$ and thereby $I_{\Theta_1}(\cdot, t_f) \subset I_{\Theta_2}(\cdot, t_f)$. As a result, if System (1) is $\Sigma-$observable for Θ_2 then it is so for Θ_1.

(7) Also one can easily prove that if System (1) is Σ_2-observable and $\Sigma_1 \subset \Sigma_2$ then it is Σ_1-observable.

(8) As a corollary of Theorem 4, System (1) is locally observable on $[t_0\ t_1]$ around $x_\star$ if there exists a real ρ such that $\mathrm{card}(\bar{B}(x_\star, \rho) \cap I(t_0, t_f)) \leq 1$.

(9) Since subset $I(t_0, t_f)$ is closed, then System (1) is locally observable around any $\bar{x}$ which does not belong to $I(t_0, t_f)$. Indeed, as the complementary of subset $I(t_0, t_f)$ is open, there will exist a ball which is centered around $\bar{x}$ and which has an empty intersection with $I(t_0, t_f)$.

Next, let us illustrate by examples.

Example 1 Let A be a matrix having an eigenvalue with positive real part, and $f(t, z) = Az$ for all $t \in \mathbb{R}_+$ and $z \in \mathbb{R}^n$. Suppose that the observability domain is given by $\Theta \doteq \mathbb{R}_+ \times \bar{B}(0, 1)$ then we get,

$$\kappa(\Theta, t_f) = [0, t_f] \times \{0\},$$

and,

$$I(t_0, t_f) = \{0\} \text{ for all } t_0.$$

Thereby the corresponding system is observable on $[t_0, t_f]$ for all t_0.

Nevertheless, if all the eigenvalues of A are of negative real part, the ball $\bar{B}(0, 1)$ is viable under A. As a result $\kappa(\Theta, t_f) = \Theta$ and $I(t_0, t_f) = \bar{B}(0, 1)$ for all t_0. Therefore system is unobservable on any horizon $[t_0\ t_f]$.

Example 2 Consider the LTI system,

$$\dot{x} = Ax, \text{ on } [t_0\ t_f], \tag{11a}$$

with output equation,

$$\theta = Cx, \text{ on } [t_0\ t_f], \tag{11b}$$

where $A \in \mathcal{L}(\mathbb{R}^n), C \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^q)$ and θ stands for a square Lebesgue integrable function with values in $\mathbb{R}^q$. The output domain can be expressed by,

$$\Theta \doteq \{(t, z) \in \mathbb{R}_+ \times \mathbb{R}^n \mid \theta(t) - Cz = 0\}.$$

It follows that,

$$(t_0, z_0) \in \kappa(\Theta, t_f) \iff (t, \exp(A(t - t_0))z_0) \in \Theta,$$

for all $t \in [t_0\ t_f]$. Hence,

$$z_0 \in I(t_0, t_f) \iff C\exp(A(t - t_0))z_0 = \theta(t),$$

for all $t \in [t_0\ t_f]$. As a result we get,

$$I(t_0, t_f) = H_{t_0}^{-1}(\{\theta\}),$$

where,

$$\begin{aligned} H_{t_0} &: \mathbb{R}^n \to L^2([t_0\ t_f], \mathbb{R}^q) \\ z_0 &\to C\exp(A(\cdot - t_0))z_0, \end{aligned}$$

which stands for a linear bounded operator. Now, by Theorem 4 (c), System (11) is observable iff $\mathrm{card}(H_{t_0}^{-1}(\{\theta\})) \leq 1$ iff H_{t_0} is injective. The latter condition is equivalent, when $\theta \in \mathrm{Im}(H_{t_0})$, to the well known rank condition,

$$\mathrm{rank}(\left[C' \mid A'C' \mid \ldots \mid A'^{(n-1)}C'\right]) = n.$$

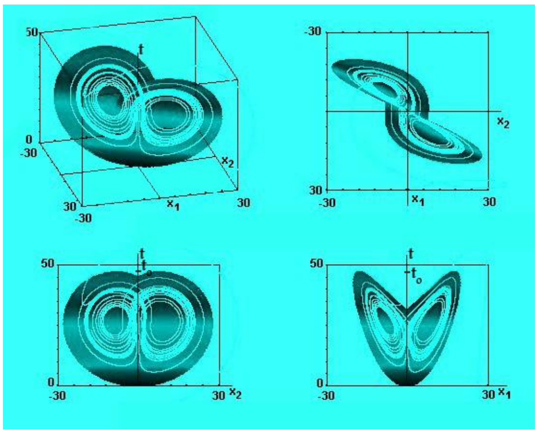

Fig. 2 These plots, adapted from Saint-Pierre and Frankowska [8], represent four views of the observability kernel (in position 1-1) of System (13), over horizon $[0, 50]$. This is observable on $[t_1\ 50]$, for all $t_1 \in \{0\} \cup [t_0\ 50]$.

Example 3 Lorenz (backward) equation consists of the following system of differential equations,

$$\begin{aligned} \dot{z}_1 &= \sigma z_1 - \sigma z_2, \\ \dot{z}_2 &= -r z_1 + z_2 + z_1 z_3 \\ \dot{z}_3 &= -z_1 z_2 + b z_3, \end{aligned} \tag{12}$$

where σ, r, b are numbers such that $\sigma > b + 1$. Denote by K the parallelepiped of Figure 2 and $K_\star$ its viability kernel under System (12), which is contained in its interior, according to [8].

Let,

$$f(t, z) = (\sigma z_1 - \sigma z_2, -r z_1 + z_2 + t z_1),$$

for all $t \geq 0, z \in \mathbb{R}^2$, then consider the system,

$$\dot{z} = f(t, z), \text{ on } [t_1\ 50], \tag{13a}$$

for $t_1 \in [0\ 50)$, with output domain given as follows,

$$\Theta \doteq \{(t, z) \in [0\ 50] \times C \mid bt = z_1 z_2\}, \tag{13b}$$

where C denotes the square $[-30\ 30] \times [-30\ 30]$. We can easily show that,

$$(t_1, z_{01}, z_{02}) \in \kappa(\Theta, 50) \Longrightarrow (z_{01}, z_{02}, t_1) \in K_\star$$

and thereby,

$$I(t_1, t_f) \subset \{(z_{01}, z_{02})|\ (z_{01}, z_{02}, t_1) \in K_\star\}.$$

By considering the subplots 2-1 and 2-2 of Figure 2, we see that the 2-D system (13) is observable on $[t_1\ 50]$, for all $t_1 \in \{0\} \cup [t_0\ 50]$, where t_0 is provided by the same figures.

4 Characterizing observability

This section is devoted to providing conditions under which the observability property holds. For that purpose, we will mainly use the preliminary results of section §2.

Let Σ be a subset of $\mathbb{R}^n$. Consider System (1) and let the multifunction $I(\cdot, t_f)$ and the function g be given as in Eqs (8) and (10), respectively.

4.1 By the principle of nests

Here, we shall use the well known nests [20] principle in metric spaces, which states that when a collection of subsets $(K_\alpha)_\alpha$ form a nest (ie. for all α, β, either $K_\alpha \subset K_\beta$ or $K_\beta \subset K_\alpha$) and satisfies $\inf_\alpha \operatorname{diam}(K_\alpha) = 0$, then their intersection contains at most one point, and the collection has a singleton intersection whenever the subsets are closed and the metrics is complete.

We begin by considering the notation,

$$z_0 \overset{f}{\rightsquigarrow} \Theta \text{ at } t,$$

to mean that System (1) has a solution which satisfies Eq. (1b) at time t. Then we consider the following assumptions,

$$\begin{array}{l}\text{either the statements :}\\ \text{"}z_0 \overset{f}{\rightsquigarrow} \Theta \text{ at } t \implies z_0 \overset{f}{\rightsquigarrow} \Theta \text{ at } s\text{"},\\ \text{or its converse hold,}\end{array} \tag{14}$$

for all $t, s \in [t_0\ t_f]$ and $z_0 \in \Sigma$, and,

$$\begin{array}{l}\text{for all } \epsilon > 0, \text{ there exists } t \in [t_0\ t_f] \text{ such that :}\\ z_1, z_2 \overset{f}{\rightsquigarrow} \Theta \text{ at } t \text{ implies } |z_1 - z_2| \leq \epsilon,\end{array} \tag{15}$$

for all $z_1, z_2 \in \Sigma$.

Then we can prove the following result.

Theorem 5

Under assumptions (14) and (15), System (1) is Σ-observable on $[t_0\ t_f]$.

Proof For each $t \in [t_0\ t_f]$ set,

$$K_t \doteq \{z_0 \in \Sigma \mid z_0 \overset{f}{\rightsquigarrow} \Theta \text{ at } t\}.$$

Thanks to Eq. (8), it follows that,

$$I(t_0, t_f) \subset \bigcap_{t_0 \leq t \leq t_f} K_t.$$

Then condition (14) is equivalent to nestedness of collection $(K_t)_{t_0 \leq t \leq t_f}$ and condition (15) means that

$$\inf_{0 \leq t \leq t_f} \operatorname{diam}(K_t) = 0.$$

Ultimately the principle of nests implies that

$$\operatorname{card}(I(t_0, t_f)) \leq \operatorname{card}(\bigcap_{t_0 \leq t \leq t_f} K_t) \leq 1,$$

and Theorem 4 (c) yields the result.

4.2 Observability and convexity

We need consider the following implication,

$$\begin{array}{l} z_1 \overset{f}{\rightsquigarrow} \Theta \text{ on } [t_1\ t_f] \\ z_2 \overset{f}{\rightsquigarrow} \Theta \text{ on } [t_2\ t_f] \end{array} \Longrightarrow \frac{z_1 + z_2}{2} \overset{f}{\rightsquigarrow} \Theta \text{ on } [\frac{t_1 + t_2}{2}\ t_f], \tag{16}$$

for z_1, z_2 in $\mathbb{R}^n$ and t_1, t_2 in $[0, t_f]$.

Then we can show the following result.

Theorem 6 Let Σ be a convex subset of $\mathbb{R}^n$ and assume that implication (16) is satisfied for all z_1, z_2 in Σ. Let $\mathcal{J} \subset \operatorname{dom}(I)$ be an open interval which contains an instant $\bar{t}$ such that System (1) is $\Sigma-$observable on horizon $[\bar{t}\ t_f]$. Then the following statements hold :

(a) System (1) is $\Sigma-$observable on $[t_0\ t_f]$ for all $t_0 \in \mathcal{J}$.

(b) There exist an additive function $\xi : \mathcal{J} \to \mathbb{R}^n$ and $y_0 \in \mathbb{R}^n$ such that for each $t_0 \in \mathcal{J}$, $\xi(t_0) + y_0$ stands for the unique state in Σ which generates the output (1b) over the horizon $[t_0\ t_f]$.

Proof First we remark that the function g of Eq. (10) has a convex graph (because $g = (1, f)'$), therefore $\operatorname{viab}_g(\Theta, t_f)$ is convex. As a result the corresponding multifunction $I(\cdot, t_f)$ of Eq. (8) is mid-convex (because it is convex). Then Theorem 1 applies for the map $I|_{\mathcal{J}}$, involving that $I(t_0)$ is a singleton for all $t_0 \in \mathcal{J}$ along with the conclusion (b).

Remark 1 The statement (a) above can be reformulated as follows : whenever, for some $\bar{t}$ in the interior of $\operatorname{dom}(I)$, System (1) is $\Sigma-$observable on $[\bar{t}\ t_f]$ then it is so from near $t_\star$.

Remark 2 We know that if an additive function is continuous at some point, it is so at all points in its domain. Thus, if the function ξ in (ii) above is continuous at some point in $\mathcal{J}$ then System (1) is continuously $\Sigma-$observable from near any instant in this interval.

Remark 3 Note that an instance where assumption (16) holds is when the output domain Θ and $\operatorname{gph}(f)$ are convex. In fact, this implies that $\operatorname{gph}(g)$ is convex and so is $\operatorname{viab}_g(\Theta, t_f)$.

4.3 Observability and monotonicity

Let us consider the following statement,

$$\begin{array}{l} z_1 \overset{f}{\rightsquigarrow} \Theta \text{ on } [\pi_1(y_1)\ t_f] \\ z_2 \overset{f}{\rightsquigarrow} \Theta \text{ on } [\pi_1(y_2)\ t_f] \end{array} \Longrightarrow \langle y_2 - y_1, z_2 - z_1 \rangle \geq 0, \tag{17}$$

for z_1, z_2 and y_1, y_2 in $\mathbb{R}^n$ such that $\pi_1(y_i) \in [0, t_f]$ for $i = 1$ or 2, where π_1 denotes the projection onto the first component.

Then we are ready to show the following result.

Theorem 7 Let Σ be a subset of $\mathbb{R}^n$ and assume that the subset $\kappa(\Theta, t_f)$ has nonempty interior and that implication (17) holds true for all z_1, z_2 in Σ. Then System (1) is $\Sigma-$observable on $[t_0\ t_f]$ for almost every t_0 in

$\mathrm{dom}(\Sigma \cap I(\cdot))$.

Proof Let I be the multifunction given by Eq. (8) and $J \doteq \Sigma \cap I(\cdot)$. Then we remark that condition (17) expresses that $J \circ \pi_1 : \mathbb{R}^n \to \mathbb{R}^n$ is monotone. Moreover, as π_1 is onto and $\kappa(\Theta, t_f)$ has nonempty interior, it follows that the interior of $\mathrm{dom}(J \circ \pi_1)$ is nonempty too.

Hence Theorem 2 yields the result.

Corollary 1 Assume that the subset $\kappa(\Theta, t_f)$ has nonempty interior and implication (17) holds true for all z_1, z_2 near a state $z_\star$, then System (1) is locally observable on $[t_0\ t_f]$ around $z_\star$, for almost every t_0 in $\mathrm{dom}(I(\cdot))$.

The next result is of great importance for our considerations because it provides a characterization of the starting instants from near which the system is continuously observable. To that end we need consider the implication below,

$$\begin{aligned} &z_1 \overset{f}{\rightsquigarrow} \Theta \text{ on } [\pi_1(y_1)\ t_f] \text{ and } z_2 \overset{f}{\rightsquigarrow} \Theta \text{ on } [\pi_1(y_2)\ t_f], \\ &\qquad\qquad\qquad \Downarrow \\ &\langle y_2 - y_1, z_2 - z_1 \rangle \geq -\langle y_2 - y_1, \tau(y_2) - \tau(y_1) \rangle, \end{aligned} \tag{18}$$

for couples $(z_1, z_2), (y_1, y_2) \in \mathbb{R}^n \times \mathbb{R}^n$, such that $\pi_1(y_i) \in [0, t_f]$ for $i = 1$ or 2 and a function $\tau : \mathbb{R}^n \to \mathbb{R}^n$.

Then we are in a position to show the following.

Theorem 8 Let $\bar{t}$ belong to $[0\ t_f), \bar{y} \doteq (\bar{t}, 0, \ldots, 0)'$ and $\bar{z}$ generate output (1b) on the horizon $[\bar{t}\ t_f]$. Then System (1) is continuously observable from near $\bar{t}$ iff the statements below are satisfied,

(a) There exist two neighborhoods $U \in \mathcal{N}(\bar{y}), V \in \mathcal{N}(\bar{z})$, and a continuous function $\tau : U \to V$, such that implication (18) holds true for all couples $(y_i, z_i) \in U \times V$ and $i = 1, 2$.

(b) Whenever a sequence $(t_q)_q$ converges to a point near $\bar{t}$, there exists a sequence $(z_q)_q$ which converges to some z near $\bar{z}$ such that $z_q \overset{f}{\rightsquigarrow} \Theta$ on $[t_q\ t_f]$, for all q.

Proof First of all, condition (b) above is only an expression of lower semicontinuity of the multifunction I near the point $(\bar{t}, \bar{z})$. As the projection π_1 is continuous, this is equivalent to lower semicontinuity of the multifunction $I \circ \pi_1$ near $(\bar{y}, \bar{z})$. Moreover statement (a) is also an explicit expression of premonotonicity of the multifunction $I \circ \pi_1$ near the point $(\bar{y}, \bar{z})$. Thereby, thanks to theorem 3, both statements (a) and (b) are equivalent to single-valuedness of the multifunction $I \circ \pi_1$ near $(\bar{y}, \bar{z})$, which is equivalent to single-valuedness of I near $(\bar{t}, \bar{z})$. This yields the result.

5 Conclusion

In this work, we have developed a new approach in order to investigate the concept of observability for systems described by ODEs. The main facts to stress can be listed as follows :

i. The set-valued framework has enabled us to :
   (a) set and deal with the observability concept in a more general context than the standard output equation.
   (b) incorporate primary knowledge of the subset where initial data must belong to.
   (c) see that local or global observability can be considered as well. In contrast, the existing results dealing with nonlinear observability are often limited to study the property near a given state.
   (d) introduce a new concept of localness with respect to starting time, and characterize it by means of premonotonicity and lower semi-continuity.
   (e) graphically check all the observability concepts, including the newly defined ones in this paper, by examining the observability kernel when it is available.
   (f) derive observability of a system out of certain conditions on the information process.

ii. Efficiency and feasibility of the method are intrinsically dependent upon the progress of investigations on the viability kernel. That also conditions the extensibility of the approach to stochastic systems and PDEs.

iii. Ultimately, an interesting open problem which might be dealt with in our set-valued framework, consists of the study of robust estimation and observer design in the presence of output uncertainties.

References

[1] Z. Bartosiewicz. Local observability of nonlinear systems[J], *Systems Control Lett.* 1995, 25: 295-298.

[2] R. Hermann, A. J. Krener. Nonlinear controllability and observability[J], *IEEE Trans. Automat. Control.* 1977, 22: 728-741.

[3] A. Isidori, Nonlinear control systems[M], third ed., Springer-verlag, London, 1995.

[4] G. Kreisselmeier, R., Engel. Nonlinear observers for autonomous Lipschitz continuous systems[J], *IEEE Trans. Automat. Control.* 2003, 48: 451 - 464.

[5] E. D. Sontag. A concept of local observability[J], *Systems Control Lett.* 1984, 5: 41-47.

[6] H. Sussmann. Single-input observability of continuous-time systems[J], *Math. Systems Theory.* 1979, 12: 371-393.

[7] J. P. Aubin. Viability theory[M], second ed., Boston: Modern Birkhäuser Classics, 2009.

[8] J. P. Aubin, P. Saint-Pierre. Viability kernels and capture basins for analyzing the dynamic behavior: Lorenz attractors, Julia sets, and Hutchinsons maps[M]// *Differential Equations, Chaos and Variational Problems*, Birkhäuser, 2008.

[9] P. Saint-Pierre. Approximation of the viability kernel[J], *Appl. Math. Optim.* 1994, 29: 187-209.

[10] N. Bonneuil. Computing the viability kernel in large state dimension[J], *J. Math. Anal. Appl.* 2006, 323: 1444-1454.

[11] H. Frankowska, M. Quincampoix. Viability kernels of differential inclusions with constraints: Algorithm and applications[J], *Mathematics of Systems, Estimation and Control.* 1991, 1: 371-388.

[12] J. Rieger. Shadowing and the viability kernel algorithm[J], *Appl Math Optim.* 2009, 60: 429-441.

[13] K. Nikodem, F. Papalini, S. Vercillo. Some representations of midconvex set-valued functions[J], *Aequationes Math.*1997, 53: 127-140.

[14] K. Nikodem, D. Popa. On single-valuedness of set-valued maps satisfying linear inclusions[J], *Banach J. Math. Anal.*2009, 3: 44-51.

[15] F. Deutch, I. Singer. On single-valuedness of convex set-valued maps[J], *Set-Valued Anal.* 1993, 1: 97-103.

[16] T., Zamfirescu. On a theorem of Deutsch and Singer[J], *Set-Valued Anal.* 2000, 8: 295-297.

[17] A. B. Levy, R. A. Poliquin. Characterizing the single-valuedness of multifunctions[J], *Set-Valued Anal.* 1997, 5: 351-364.

[18] E. H. Zarantonello. Dense single-valuedness of monotone operators[J], *Israel J. Math.* 1973, 15: 158-166.

[19] V. Deimling. *Multivalued differential equations*[M], Berlin: Walter de Gruyter, 1992.

[20] M. O. Searcóid. *Metric spaces*[M], London: Springer, 2007.

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